The classification of root systems

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Definition of the root system

Definition
Let $\mathbb{E} \cong \mathbb{R}^n$ be a real vector space. A finite subset $R \subset \mathbb{E}$ is called root system if

1. span $R = \mathbb{E}$, $0 \notin R$,
2. $\pm \alpha \in R$ are the only multiples of $\alpha \in R$,
3. $R$ is invariant under reflections $s_\alpha$ in hyperplanes orthogonal to any $\alpha \in R$,
4. if $\alpha, \beta \in R$, then $n_{\beta\alpha} = 2 \langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$.

The elements of $R$ are called roots. The rank of the root system is the dimension of $\mathbb{E}$.
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$$n_{\beta \alpha} = \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$ 

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Restrictions

Projection

\[
\text{proj}_\alpha \beta = \alpha \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{1}{2} n_{\beta \alpha} \alpha
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Angles

$$n_{\beta \alpha} = 2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2 \frac{||\beta||}{||\alpha||^2} \frac{||\alpha|| \cos \theta}{||\alpha||} = 2 \frac{||\beta||}{||\alpha||} \cos \theta \in \mathbb{Z}$$

$$n_{\beta \alpha} \cdot n_{\alpha \beta} = 4 \cos^2 \theta \in \mathbb{Z}$$

$$4 \cos^2 \theta \in \{0, 1, 2, 3, 4\}$$
Angles

\[4 \cos^2 \theta \in \{0, 1, 2, 3\}\], or \(\cos \theta \in \pm \left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}\]
Examples in rank 2

Root system $A_1 \times A_1$

(decomposable)
Examples in rank 2

Root system $A_2$
Examples in rank 2

Root system $B_2$
Examples in rank 2

Root system $G_2$
Consider a vector $d$, such that $\forall \alpha \in R : \langle \alpha, d \rangle \neq 0$. Define $R^+(d) = \{ \alpha \in R | \langle \alpha, d \rangle > 0 \}$. Then $R = R^+(d) \cup R^-(d)$, where $R^-(d) = -R^+(d)$. 

Definition: A root $\alpha$ is called positive if $\alpha \in R^+(d)$ and negative if $\alpha \in R^-(d)$. 

Definition: A positive root $\alpha \in R^+(d)$ is called simple if it is not a sum of two other positive roots. 

Definition: The set of all simple roots of a root system $R$ is called basis of $R$. 

Positive roots and simple roots
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Properties of simple roots

**Definition**
The hyperplanes orthogonal to $\alpha \in R$ cut the space $E$ into open, connected regions called *Weyl chambers*.

**Lemma** There is a one-to-one correspondence between bases and Weyl chambers.

**Definition** The group generated by reflections $s_{\alpha}$ is called the *Weyl group*.

**Lemma** Any two bases of a given root system $R \subset E$ are equivalent under the action of the Weyl group.

**Lemma** The root system $R$ can be uniquely reconstructed from its basis.
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Conclusion
Since $4 \cos^2 \theta \in \{0, 1, 2, 3\}$, it means that $\theta \in \{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\}$. 
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Definition
The Coxeter graph of a root system $R$ is a graph that has one vertex for each simple root of $R$ and every pair $\alpha, \beta$ of distinct vertices is connected by $n_{\alpha\beta} \cdot n_{\beta\alpha} = 4 \cos^2 \theta \in \{0, 1, 2, 3\}$ edges.
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Definition
The Dynkin diagram of a root system is its Coxeter graph with arrow attached to each double and triple edge pointing from longer root to shorter root.
Admissible diagrams

Definition
A set of $n$ unit vectors $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{E}$ is called an *admissible configuration* if:

1. $v_i$’s are linearly independent and span $\mathbb{E}$,
2. if $i \neq j$, then $\langle v_i, v_j \rangle \leq 0$,
3. and $4 \langle v_i, v_j \rangle^2 = 4 \cos^2 \theta \in \{0, 1, 2, 3\}$. 

Note
The set of normalized simple roots of any root system is an admissible configuration (they are linearly independent, span the whole space, and have specific angles between them).
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Definition
Coxeter graph of an admissible configuration is admissible diagram.
Irreducibility

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If a root system is not decomposable, it is called irreducible.
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If a root system is not decomposable, it is called *irreducible*.

Lemma
*The root system is irreducible if and only if its base is irreducible.*
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 Lemma
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Conclusion
It means, the set of simple roots of an irreducible root system can not be decomposed into mutually orthogonal subsets. Hence the corresponding Coxeter graph will be *connected*. Thus, to classify all irreducible root systems, it is enough to consider only connected admissible diagrams.
Classification theorem

Theorem

*The Dynkin diagram of an irreducible root system is one of:*

- $A_n$: \[n \leq 1\]
- $B_n$: \[n \leq 2\]
- $C_n$: \[n \leq 3\]
- $D_n$: \[n \leq 4\]
- $E_6$
- $E_7$
- $E_8$
- $F_4$
- $G_2$
Step 1

Claim: Any subdiagram of an admissible diagram is also admissible.
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If the set \( \{v_1, v_2, \ldots, v_n\} \) is an admissible configuration, then clearly any subset of it is also an admissible configuration (in the space it spans). The same holds for admissible diagrams.
Step 2

Claim: A connected admissible diagram is a tree.
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Define \( v = \sum_{i=1}^{n} v_i \) (\( v \neq 0 \)). Then

\[
0 < \langle v, v \rangle = \sum_{i=1}^{n} \langle v_i, v_i \rangle + \sum_{i<j} 2 \langle v_i, v_j \rangle = n + \sum_{i<j} 2 \langle v_i, v_j \rangle.
\]

If \( v_i \) and \( v_j \) are connected, then

\[
2 \langle v_i, v_j \rangle \in \left\{ -1, -\sqrt{2}, -\sqrt{3} \right\}
\]

In particular, \( 2 \langle v_i, v_j \rangle \leq -1 \). It means, the number of terms in the sum and hence the number of edges can not exceed \( n - 1 \).
Step 3

**Claim:** No more than three edges (counting multiplicities) can originate from the same vertex.
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Let $v_1, v_2, \ldots, v_k$ be connected to $c$, then $\langle v_i, v_j \rangle = \delta_{ij}$. Let $v_0 \neq 0$ be the normalized projection of $c$ to the orthogonal complement of $v_i$’s. Then $\{v_0, v_1, v_2, \ldots, v_k\}$ is an orthonormal basis and:

$$c = \sum_{i=0}^{k} \langle c, v_i \rangle v_i.$$

Since $\langle c, c \rangle = \sum_{i=0}^{k} \langle c, v_i \rangle^2 = 1$ and $\langle c, v_0 \rangle \neq 0$, then

$$\sum_{i=1}^{k} 4 \langle c, v_i \rangle^2 < 4,$$

where $4 \langle c, v_i \rangle^2$ is the number of edges between $c$ and $v_i$. 
Claim: The only connected admissible diagram containing a triple edge is

\[ G_2 \]

This follows from the previous step. From now on we will consider only diagrams with single and double edges.
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Step 5

Claim: Any simple chain \(v_1, v_2, \ldots, v_k\) can be replaced by a single
vector \(v = \sum_{i=1}^{k} v_i\).
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**Claim:** Any simple chain $v_1, v_2, \ldots, v_k$ can be replaced by a single vector $v = \sum_{i=1}^{k} v_i$.

Vector $v$ is a unit vector, since $2 \langle v_i, v_j \rangle = -\delta_{i+1,j}$ and therefore

$$\langle v, v \rangle = k + \sum_{i<j} 2 \langle v_i, v_j \rangle = k + \sum_{i=1}^{k-1} 2 \langle v_i, v_{i+1} \rangle = k - (k - 1) = 1.$$

If $u$ is not in the chain, then it can be connected to at most one vertex in the chain (let it be $v_j$). Then

$$\langle u, v \rangle = \sum_{i=1}^{k} \langle u, v_i \rangle = \langle u, v_j \rangle$$

and $u$ remains connected to $v$ in the same way. Therefore the obtained diagram is also admissible and connected.
Step 6

Claim: A connected admissible diagram has none of the following subdiagrams:

- [Diagram showing the subdiagrams described in the claim]
**Step 6**

**Claim:** A connected admissible diagram has none of the following subdiagrams:

![Diagrams showing subdiagrams](image)

**Conclusion**

It means that a connected admissible diagram can contain at most one double edge and at most one branching, but not both of them simultaneously.
Step 7

**Claim:** There are only three types of connected admissible diagrams:

**T1:** a simple chain,

**T2:** a diagram with a double edge,

**T3:** a diagram with branching.

![Diagram of T1](image1)

![Diagram of T2](image2)

![Diagram of T3](image3)
Claim: The admissible diagram of type T1 corresponds to the Dynkin diagram $A_n$, where $n \geq 1$. 

\[ A_n \begin{array}{cccccccccc}
  \circ & \quad & \circ & \quad & \cdots & \quad & \circ & \quad & \circ & \quad & \circ \\
 & & & & & & & & & \\
\end{array} \\
(n \leq 1)
Claim: The admissible diagrams of type $T2$ are $F_4$, $B_n$, and $C_n$. 
Step 9

Claim: The admissible diagrams of type $T2$ are $F_4$, $B_n$, and $C_n$.

Define $u = \sum_{i=1}^{p} i \cdot u_i$. Since $2 \langle u_i, u_{i+1} \rangle = -1$ for $1 \leq i \leq p - 1$,

$$
\langle u, u \rangle = \sum_{i=1}^{p} i^2 \langle u_i, u_i \rangle + \sum_{i<j} ij \cdot 2 \langle u_i, u_j \rangle = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i + 1)
$$

$$
= p^2 - \sum_{i=1}^{p-1} i = p^2 - \frac{p(p - 1)}{2} = \frac{p(p + 1)}{2}.
$$

Similarly, $v = \sum_{j=1}^{q} j \cdot v_j$ and $\langle v, v \rangle = q(q + 1)/2$. From $\langle u, v \rangle = pq \langle u_p, v_q \rangle$ and $4 \langle u_p, v_q \rangle^2 = 2$ we get $\langle u, v \rangle^2 = p^2 q^2 / 2$. From Cauchy-Schwarz inequality $\langle u, v \rangle^2 < \langle u, u \rangle \langle v, v \rangle$ we get

$$
\frac{p^2 q^2}{2} < \frac{p(p + 1)}{2} \cdot \frac{q(q + 1)}{2}.
$$
Step 10 (continued)

Since $p, q \in \mathbb{Z}_+$, we get $2pq < (p + 1)(q + 1)$ or simply $(p - 1)(q - 1) < 2$. 

$p = q = 2 \text{ and } p = 1 \text{ and } q \text{ is arbitrary (or vice versa)}$
Step 10 (continued)

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$F_4$
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$p = q = 2$

\[ F_4 \]

$p = 1$ and $q$ is arbitrary (or vice versa)

\[ B_n \]

\[ C_n \]

(n\leq3)
Claim: The admissible diagrams of type T3 are $D_n, E_6, E_7, E_8$. 
**Claim:** The admissible diagrams of type T3 are $D_n$, $E_6$, $E_7$, $E_8$.

Define $u = \sum_{i=1}^{p-1} i \cdot u_i$, $v = \sum_{j=1}^{q-1} j \cdot v_j$, and $w = \sum_{k=1}^{r-1} k \cdot w_k$. Let $u'$, $v'$, and $w'$ be the corresponding unit vectors. Then

$$1 = \langle c, c \rangle > \langle c, u' \rangle^2 + \langle c, v' \rangle^2 + \langle c, w' \rangle^2.$$ 

Since $\langle c, u_i \rangle^2 = 0$ unless $i = p - 1$ and $4 \langle c, u_{p-1} \rangle^2 = 1$, we have

$$\langle c, u \rangle^2 = \sum_{i=1}^{p-1} i^2 \langle c, u_i \rangle^2 = (p - 1)^2 \langle c, u_{p-1} \rangle^2 = \frac{(p - 1)^2}{4}.$$ 

We already know that $\langle u, u \rangle = p(p - 1)/2$, therefore

$$\langle c, u' \rangle^2 = \frac{\langle c, u \rangle^2}{\langle u, u \rangle} = \frac{(p - 1)^2}{4} \cdot \frac{2}{p(p - 1)} = \frac{p - 1}{2p} = \frac{1}{2} \left( 1 - \frac{1}{p} \right).$$
If we do the same for \( v \) and \( w \), we get

\[
2 > (1 - 1/p) + (1 - 1/q) + (1 - 1/r)
\]

or simply

\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1, \quad p, q, r \geq 2.
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Step 10 (Continued)

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We can assume that $p \geq q \geq r \geq 2$. There is no solution with $r \geq 3$, since then the sum can not exceed 1. Therefore we have to take $r = 2$. If we take $q = 2$ as well, then any $p$ suits, but for $q = 3$ we have $1/q + 1/r = 5/6$ and we can take only $p < 6$. There are no solutions with $q \geq 4$, because then the sum is at most 1.
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or simply

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<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>Dynkin diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>any</td>
<td>2</td>
<td>2</td>
<td>$D_n$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>$E_6$</td>
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<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>$E_7$</td>
</tr>
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<td>5</td>
<td>3</td>
<td>2</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>
End of proof

Q.E.D.
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Theorem

For each Dynkin diagram we have found there indeed is an irreducible root system having the given diagram.